

# Elasticity of Fractal Material by Continuum Model with Non-Integer Dimensional Space

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## Abstract

Using a generalization of vector calculus for space with non-integer dimension, we consider elastic properties of fractal materials. Fractal materials are described by continuum models with non-integer dimensional space. A generalization of elasticity equations for non-integer dimensional space, and its solutions for equilibrium case of fractal materials are suggested. Elasticity problems for fractal hollow ball and cylindrical fractal elastic pipe with inside and outside pressures, for rotating cylindrical fractal pipe, for gradient elasticity and thermoelasticity of fractal materials are solved.

PACS: 62.20.Dc; 81.40.Jj; 45.10.Hj

## 1 Introduction

Fractals are measurable metric sets with non-integer Hausdorff dimensions [1, 2]. The main characteristic of fractal set is non-integer Hausdorff dimension that should be observed on all scales. The Hausdorff dimension is a local property, i.e. this dimension characterize (measure) property of a set of distributed points in the limit of a vanishing diameter, which is used to cover subset of the points. By definition the Hausdorff dimension requires the diameter of the covering sets to vanish. In general, real materials have a characteristic smallest length scale  $R_0$  such as the radius of a particle such as atom or molecule. In fractal materials the fractal structure cannot be observed on all scales but only those for which  $R > R_0$ , where  $R_0$  is the characteristic scale of the particles. For real materials, a non-integer mass dimension can be used instead of Hausdorff dimension. The mass dimension described how the mass of a medium region scales with the size of this region, where we assume unchanged density. For many cases, we have an asymptotic relation between the mass  $M(W)$  of a ball region  $W$  of material, and the radius  $R$  of this ball. The mass of fractal material satisfies a power-law relation  $M(W) \sim R^D$ . The parameter  $D$  is called the non-integer mass dimension of fractal material. This parameter does not depend on the shape of the region  $W$ , or on whether the packing of sphere of radius  $R_0$  is close packing, a random packing or a porous packing with a uniform distribution of holes.

Therefore fractal material can be considered as a medium with non-integer mass dimension. Although, the non-integer dimension does not reflect completely the geometrical and dynamical properties of the fractal materials, it nevertheless permits a number of important conclusions about the behavior of materials. It allows us to use effective models that take into account non-integer dimensions.

We can distinguish the following approaches to formulate models of fractal materials:

1) Approach based on methods of "Analysis on fractals" [3, 4, 5, 6, 7, 8] can be considered as the most rigorous approach to describe fractal materials. Unfortunately a possibility of application of the "Analysis on fractals" to solve real problems of fractal material now is very limited due to weak development of this area of mathematics.

2) To describe fractal material we can apply special continuum models suggested in [9, 10, 11, 12] and then developed in the works [13]-[20] and [21]. These models can be called the fractional-integral continuum models. In this approach we use integrations of non-integer orders, and two different notions such as density of states and distribution function [21]. The order of the fractional integrals is equal to mass dimension of fractal materials. The kernels of these integrals are defined by the power-law type of density of states.

3) Fractional derivatives of non-integer orders are used to describe some properties of fractal materials. This approach has been suggested in papers [22, 23, 24], where so-called local fractional derivatives are used, and then developed in the works [25, 26, 27, 28, 29, 30, 31]. These models can be called the fractional-differential models.

4) Fractal materials can be described by using the theory of integration and differentiation for non-integer dimensional spaces [32, 33, 34]. Fractal materials are described as continuum in non-integer dimensional spaces. The dimension of the spaces are equal to the mass dimensions of fractal materials.

Unfortunately there are not enough differential equations that are solved for various problems for fractal materials in the framework of the fractional-differential model and by methods of "Analysis on fractals".

The fractional-integral continuum models are used to solve differential equations for various problems of elasticity of fractal materials [15, 16, 17, 18, 19, 20], and thermoelasticity of fractal materials [70, 71].

Continuum models with non-integer dimensional spaces are not currently used to describe elasticity of fractal materials. In this paper, we consider approach based on the non-integer dimensional space to describe elasticity of isotropic fractal materials. The main difference of the continuum models with non-integer dimensional spaces and fractional-integral continuum models suggested in [9, 10, 11, 12, 21] may be reduced to the following: (a) Arbitrariness in the choice of the numerical factor in the density of states is fixed by the equation of the volume of non-integer dimensional ball region. (b) In the fractional-integral continuum models the differentiations are integer orders whereas the integrations are non-integer orders. In the continuum models with non-integer dimensional spaces the integrations and differentiations are defined for the spaces with non-integer dimensions. The power law  $M \sim R^D$  can be naturally derived by using the integrations in non-integer dimensional space [32], where the dimension of

this space is equal to the mass dimension of fractal material.

A vector calculus for non-integer dimensional space proposed in this paper allows us to use continuum models, which are based on non-integer dimensional space, to describe fractal materials. This is due to the fact that although the non-integer dimension does not reflect all geometrical and dynamical properties of the fractal materials, it nevertheless allows us to get important results about the behavior of fractal materials. Therefore continuum models with non-integer dimensional spaces can describe a wide class of fractal materials.

Integration over non-integer dimensional spaces are actively used in the theory of critical phenomena and phase transitions in statistical physics [35, 36], and in the dimensional regularization of ultraviolet divergences in quantum field theory [37, 38, 32]. The axioms for integrations in non-integer dimensional space are proposed in [39, 33] and this type of integration is considered in the book by Collins [32] for rotationally covariant functions. In the paper [33] a mathematical basis of integration on non-integer dimensional space is given, and a generalization of the Laplace operator for non-integer dimensional spaces is suggested. Using a product measure approach, the Stillinger's methods [33] has been generalized by Palmer and Stavrinou [34] for multiple variables case with different degrees of confinement in orthogonal directions. The scalar Laplace operators suggested by Stillinger in [33] and by Palmer, Stavrinou in [34] for non-integer dimensional spaces, have successfully been used for effective descriptions in physics and mechanics. The Stillinger's form of Laplacian for the Schrödinger equation in non-integer dimensional space is used by He [40, 41, 42] to describe a measure of the anisotropy and confinement by the effective non-integer dimensions. Quantum mechanical models with non-integer (fractional) dimensional space has been discussed in [33, 34, 43, 44, 45, 46, 47] and [48, 49, 50, 51]. Recent progress in non-integer dimensional space approach also includes description of the fractional diffusion processes in non-integer dimensional space in [52], and the electromagnetic fields in non-integer dimensional space in [53, 54, 55] and [56, 57, 58, 59].

Unfortunately, in the articles [33, 34] are proposed only the second order differential operators for scalar fields in the form of the scalar Laplacian for the non-integer dimensional space. A generalization of the vector Laplacian [61] for the non-integer dimensional space is not suggested. The first order operators such as gradient, divergence, curl operators, and the vector Laplacian are not considered in [33, 34] also. In the work [60] the gradient, divergence, and curl operators are suggested only as approximations of the square of the Laplace operator. Consideration only the scalar Laplacian in non-integer dimensional space approach greatly restricts us in application of continuum models with non-integer dimensional space for fractal materials and material. For example, we cannot use the Stillinger's form of Laplacian for the displacement vector field  $\mathbf{u}(\mathbf{r}, t)$  in theory of elasticity and thermoelasticity of fractal materials, for the velocity vector field  $\mathbf{v}(\mathbf{r}, t)$  in hydrodynamics of fractal fluids, for electric and magnetic vector fields in electrodynamics of fractal media in the framework non-integer dimensional space approach.

In this paper, we define the first and second orders differential vector operations such as gradient, divergence, the scalar and vector Laplace operators for non-integer dimensional space. In order to derive the vector differential operators in non-integer dimensional space we use

the method of analytic continuation in dimension. For simplification we consider rotationally covariant scalar and vector functions that are independent of angles. It allows us to reduce differential equations in non-integer dimensional space to ordinary differential equations with respect to  $r$ . The proposed operators allows us to describe fractal materials to describe processes in the framework of continuum models with non-integer dimensional spaces. In this paper we solve elasticity problems for fractal hollow ball with inside and outside pressures, for cylindrical fractal elastic pipe with inside and outside pressures, and for rotating cylindrical fractal pipe, for gradient elasticity and thermoelasticity of fractal materials.

## 2 Differential and integral operators in non-integer dimensional space

To derive equations for vector differential operators in non-integer dimensional space, we use equations for the differential operators in the spherical (and cylindrical) coordinates in  $\mathbb{R}^n$  for arbitrary  $n$  to highlight the explicit relations with dimension  $n$ . Then the vector differential operators for non-integer dimension  $D$  can be defined by continuation in dimension from integer  $n$  to non-integer  $D$ . To simplify we will consider only scalar fields  $\varphi$  and vector fields  $\mathbf{u}$  that are independent of angles

$$\varphi(\mathbf{r}) = \varphi(r), \quad \mathbf{u}(\mathbf{r}) = \mathbf{u}(r) = u_r \mathbf{e}_r,$$

where  $r = |\mathbf{r}|$  is the radial distance,  $\mathbf{e}_r = \mathbf{r}/r$  is the local orthogonal unit vector in the directions of increasing  $r$ , and  $u_r = u_r(r)$  is the radial component of  $\mathbf{u}$ . We will work with rotationally covariant functions only. This simplification is analogous to the simplification for definition of integration over non-integer dimensional space described in Section 4 of the book [32].

### 2.1 Vector differential operators for spherical and cylindrical cases

Explicit definitions of differential operators for non-integer dimensional space can be obtained by using continuation from integer  $n$  to arbitrary non-integer  $D$ . We note that the same expressions can be obtained by using the integration in non-integer dimensional space and the correspondent Gauss's theorem.

We define the differential vector operations such as gradient, divergence, the scalar and vector Laplacian for non-integer dimensional space. For simplifications, we assume that the vector field  $\mathbf{u} = \mathbf{u}(\mathbf{r})$  be radially directed and the scalar and vector fields  $\varphi(\mathbf{r})$ ,  $\mathbf{u}(\mathbf{r})$  are not dependent on the angles.

The divergence in non-integer dimensional space for the vector field  $\mathbf{u} = \mathbf{u}(r)$  is

$$\text{Div}_r^D \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{D-1}{r} u_r. \quad (1)$$

The gradient in non-integer dimensional space for the scalar field  $\varphi = \varphi(r)$  is

$$\text{Grad}_r^D \varphi = \frac{\partial \varphi}{\partial r} \mathbf{e}_r. \quad (2)$$

The scalar Laplacian in non-integer dimensional space for the scalar field  $\varphi = \varphi(r)$  is

$${}^S \Delta_r^D \varphi = \text{Div}_r^D \text{Grad}_r^D \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{D-1}{r} \frac{\partial \varphi}{\partial r}. \quad (3)$$

The vector Laplacian in non-integer dimensional space for the vector field  $\mathbf{u} = u_r(r) \mathbf{e}_r$  is

$${}^V \Delta_r^D \mathbf{u} = \text{Grad}_r^D \text{Div}_r^D \mathbf{u} = \left( \frac{\partial^2 u_r}{\partial r^2} + \frac{D-1}{r} \frac{\partial u_r}{\partial r} - \frac{D-1}{r^2} u_r \right) \mathbf{e}_r. \quad (4)$$

If  $D = n$ , equations (1-4) give the well-known formulas for integer dimensional space  $\mathbb{R}^n$ .

We can consider materials with axial symmetry, where the fields  $\varphi(r)$  and  $\mathbf{u}(r) = u_r(r) \mathbf{e}_r$  are also axially symmetric. Let the  $Z$ -axis be directed along the axis of symmetry. Therefore we use a cylindrical coordinate system.

The divergence in non-integer dimensional space for the vector field  $\mathbf{u} = \mathbf{u}(r)$  is

$$\text{Div}_r^D \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{D-2}{r} u_r. \quad (5)$$

The gradient in non-integer dimensional space for the scalar field  $\varphi = \varphi(r)$  is

$$\text{Grad}_r^D \varphi = \frac{\partial \varphi}{\partial r} \mathbf{e}_r. \quad (6)$$

The scalar Laplacian in non-integer dimensional space for the scalar field  $\varphi = \varphi(r)$  is

$${}^S \Delta_r^D \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{D-2}{r} \frac{\partial \varphi}{\partial r}. \quad (7)$$

The vector Laplacian in non-integer dimensional space for the vector field  $\mathbf{u} = v(r) \mathbf{e}_r$  is

$${}^V \Delta_r^D \mathbf{u} = \left( \frac{\partial^2 u_r}{\partial r^2} + \frac{D-2}{r} \frac{\partial u_r}{\partial r} - \frac{D-2}{r^2} u_r \right) \mathbf{e}_r. \quad (8)$$

Equations (5-8) can be easily generalized for the case  $\varphi = \varphi(r, z)$  and  $\mathbf{u}(r, z) = u_r(r, z) \mathbf{e}_r + u_z(r, z) \mathbf{e}_z$ . In this case the curl operator for  $\mathbf{u}(r, z)$  is different from zero, and

$$\text{Curl}_r^D \mathbf{u} = \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \mathbf{e}_\theta. \quad (9)$$

Equations (1) - (8) with  $D = 3$  and (9) give the well-known expressions for the gradient, divergence, curl operator, scalar and vector Laplacian operators

The proposed operators for  $0 < D < 3$  allows us to reduce non-integer dimensional vector differentiations (1) - (4) and (5) - (8) to derivatives with respect to  $r = |\mathbf{r}|$ . It allows us to reduce partial differential equations for fields in non-integer dimensional space to ordinary differential equations with respect to  $r$ .

For a function  $\varphi = \varphi(r, \theta)$  of radial distance  $r$  and related angle  $\theta$  measured relative to an axis passing through the origin, the scalar Laplacian in a non-integer dimensional space proposed by Stillinger [33] is

$${}^{St}\Delta^D = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin^{D-2} \theta} \frac{\partial}{\partial \theta} \left( \sin^{D-2} \theta \frac{\partial}{\partial \theta} \right), \quad (10)$$

where  $D$  is the dimension of space ( $0 < D < 3$ ), and the variables  $r \geq 0$ ,  $0 \leq \theta \leq \pi$ . Note that  $({}^{St}\Delta^D)^2 \neq {}^{St}\Delta^{2D}$ . If the function depends on radial distance  $r$  only ( $\varphi = \varphi(r)$ ), then

$${}^{St}\Delta^D \varphi(r) = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial \varphi(r)}{\partial r} \right) = \frac{\partial^2 \varphi(r)}{\partial r^2} + \frac{D-1}{r} \frac{\partial \varphi(r)}{\partial r}. \quad (11)$$

It is easy to see that the Stillinger's form of Laplacian  ${}^{St}\Delta^D$  for radial scalar functions  $\varphi(\mathbf{r}) = \varphi(r)$  coincides with the scalar Laplacian  ${}^S\Delta_r^D$  defined by (3), i.e.,

$${}^{St}\Delta^D \varphi(r) = {}^S\Delta_r^D \varphi(r). \quad (12)$$

The Stillinger's Laplacian can be applied for scalar fields only. It cannot be used to describe vector fields  $\mathbf{u} = u_r(r) \mathbf{e}_r$  because this Laplacian for  $D = 3$  is not equal to the usual vector Laplacian for  $\mathbb{R}^3$ ,

$${}^{St}\Delta^3 \mathbf{u}(r) \neq \Delta \mathbf{u}(r) = \left( \frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r} \frac{\partial u_r}{\partial r} - \frac{2}{r^2} u_r \right) \mathbf{e}_r. \quad (13)$$

The gradient, divergence, curl operator and vector Laplacian are not considered by Stillinger in paper [33].

## 2.2 Integration over non-integer dimensional space

Integration for non-integer values of dimension  $D$  is defined by continuation in  $D$  [38, 32]. The following properties suggested in [39] for integrals in  $D$ -dimensional space are necessary for applications [32]:

a) Linearity:

$$\int \left( a f_1(\mathbf{r}) + b f_2(\mathbf{r}) \right) d^D \mathbf{r} = a \int f_1(\mathbf{r}) d^D \mathbf{r} + b \int f_2(\mathbf{r}) d^D \mathbf{r}, \quad (14)$$

where  $a$  and  $b$  are arbitrary real numbers.

b) Translational invariance:

$$\int f(\mathbf{r} + \mathbf{r}_0) d^D \mathbf{r} = \int f(\mathbf{r}) d^D \mathbf{r} \quad (15)$$

for any vector  $\mathbf{r}_0$ .

c) Scaling property:

$$\int f(\lambda \mathbf{r}) d^D \mathbf{r} = \lambda^{-D} \int f(\mathbf{r}) d^D \mathbf{r} \quad (16)$$

for any positive  $\lambda$ .

Note that linearity is true of any integration, while translation and rotation invariance are basic properties of a Euclidean space. The scaling property embodies the  $D$ -dimensionality. Not only the above three axioms are necessary, but they also ensure that integration is unique, aside from an overall normalization [39]. These properties must be used in order to have non-integer dimensional integrations [32]. These properties are natural in application of dimensional regularization to quantum field theory [32].

In general, we can consider any functions of the components of its vector argument  $\mathbf{r}$ . However, we do not know the meaning of the components of a vector in non-integer dimensions. In this paper, we will work with rotationally covariant functions for simplification. So we will assume that  $f$  is a scalar or vector function only of scalar products of vectors or of length of vectors. For example, in the elasticity theory, we consider the case, where the displacement vector  $\mathbf{u}(\mathbf{r})$ , is independent of the angles  $\mathbf{u}(\mathbf{r}) = \mathbf{u}(r)$ , where  $r = |\mathbf{r}|$ . The integration defined by equation (17) satisfies the properties (14) - (16).

The non-integer dimensional integration for scalar functions  $f(\mathbf{r}) = f(|\mathbf{r}|)$  can be defined in terms of ordinary integration by the equation

$$\int d^D \mathbf{r} f(\mathbf{r}) = \int_{\Omega_{D-1}} d\Omega_{D-1} \int_0^\infty dr r^{D-1} f(r), \quad (17)$$

where

$$\int_{\Omega_{D-1}} d\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)} = S_{D-1}. \quad (18)$$

Equation (18) with integer  $D = n$  gives the well-known area  $S_{n-1}$  of  $(n-1)$ -sphere with unit radius.

As a result, we have [32] the explicit definition of the continuation of integration from integer  $n$  to arbitrary fractional  $D$  in the form

$$\int d^D \mathbf{r} f(|\mathbf{r}|) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty dr r^{D-1} f(r). \quad (19)$$

This equation reduced non-integer dimensional integration to ordinary integration. Therefore the linearity and translation invariance follow from linearity and translation invariance of ordinary integration. The scaling and rotation covariance are explicit properties of the definition.

As an example of applications of equation (19), we can consider non-integer dimensional integration for the function

$$f(\mathbf{r}^2) = \frac{\mathbf{r}^2 + a}{\mathbf{r}^2 + b}, \quad (20)$$

where  $a$  and  $b$  are real numbers. The integral for (20) can be explicitly computed

$$\int d^D \mathbf{r} \frac{r^2 + a}{r^2 + b} = (\pi b)^{D/2} (a/b - 1) \Gamma(1 - D/2). \quad (21)$$

The other example is the non-integer dimensional integration is

$$\int d^D \mathbf{r} \frac{r^{2\alpha}}{(r^2 + a^2)^\beta} = \frac{\Gamma(\alpha + D/2) \Gamma(\beta - \alpha - D/2)}{\Gamma(D/2) \Gamma(\beta)} \pi^{D/2} a^{D+2\alpha-2\beta}, \quad (22)$$

where  $r = |\mathbf{r}|$ .

### 3 Mass and moment of inertia for fractal materials

#### 3.1 Mass of fractal materials

Fractal materials can be characterized by the relation between the mass  $M(W)$  of a region  $W$  of fractal material, and the size  $R$  of the region containing this mass:

$$M_D(W) = M_0 \left( \frac{R}{R_0} \right)^D, \quad R/R_0 \gg 1. \quad (23)$$

The parameter  $D$  is called the non-integer mass dimension of fractal material. The parameter  $D$ , does not depend on the shape of the region  $W$ , or on whether the packing of sphere of radius  $R_0$  is close packing, a random packing or a porous packing with a uniform distribution of holes. The cornerstone of fractal materials is the non-integer mass dimension. The mass dimension of real fractal materials can be measured by box-counting method, which means drawing a box of size  $R$  and counting the mass inside.

The fractality of material means than the mass of the region  $W \subset \mathbb{R}^3$  increases more slowly than the 3-dimensional volume of this region. For the ball region of the fractal medium, this property can be described by the power law  $M_D(W) \sim R^D$ , where  $R$  is the radius of the ball.

Fractal material is called homogeneous if the power law  $M_D(W) \sim R^D$  does not depend on the translation of the region. The homogeneity property of the material can be formulated in the form: For all two regions  $W_1$  and  $W_2$  of the homogeneous fractal material with the equal volumes  $V_D(W_1) = V_D(W_2)$ , the masses of these regions are equal  $M_D(W_1) = M_D(W_2)$ .

The power law (23) can be naturally derived by using the integration in the non-integer dimensional space such that the space dimensions is equal to the mass dimension of the material.

The mass of the region  $W$  of fractal material in  $W$  can be calculated by the integral in non-integer dimensional space

$$M_D(W) = \int_W \rho(\mathbf{r}) d^D \mathbf{r}, \quad (24)$$



where  $\mathbf{r}$  is dimensionless vector variable. For a ball region  $W$  with radius  $R$  and density  $\rho(\mathbf{r}) = \rho_0 = \text{const}$ , we get the mass is defined by

$$M_D(W) = \rho_0 V_D = \frac{\pi^{D/2} \rho_0}{\Gamma(D/2 + 1)} R^D. \quad (25)$$

This equation define the mass of the fractal homogeneous ball. For  $D = 3$ , equation (25) gives the well-known equation for mass of non-fractal ball  $M_3 = (4\rho_0\pi)/3R^3$  because  $\Gamma(3/2) = \sqrt{\pi}/2$  and  $\Gamma(z + 1) = z \Gamma(z)$ .

### 3.2 Moment of inertia of fractal materials

Let us consider a calculation of scalar moment of inertia  $I(t)$ , which is used when the axis of rotation is known. The scalar moment of inertia of a rigid body with density  $\rho'(\mathbf{r}', t)$  with respect to a given axis is defined by the volume integral

$$I'(t) = \int_W \rho'(\mathbf{r}', t) \mathbf{r}'_{\perp}^2 d^3\mathbf{r}', \quad (26)$$

where  $(\mathbf{r}')_{\perp}^2$  is the perpendicular distance from the axis of rotation, and  $dV'_3 = dx'_1 dx'_2 dx'_3$ . We note that SI units of  $I'_{kl}$  is  $kg \cdot m^2$ .

To generalize equation (26) for non-integer dimensional space, we should represent this equation through the dimensionless coordinate variables. We can introduce the dimensionless values  $x_k = x'_k/R_0$ ,  $\mathbf{r} = \mathbf{r}'/R_0$ , where  $R_0$  is a characteristic scale, and the density  $\rho(\mathbf{r}, t) = R_0^3 \rho'(\mathbf{r} R_0, t)$ . SI units of  $\rho$  is  $kg$ . We define the following moments of inertia  $I(t) = R_0^{-2} I'(t)$ . As a result, we obtain

$$I = \int_{W_3} \rho(\mathbf{r}) \mathbf{r}_{\perp}^2 d^3\mathbf{r}, \quad (27)$$

where  $x_k$  ( $k = 1, 2, 3$ ) and  $\mathbf{r}$  are dimensionless. We note that SI units of  $I_{kl}$  is  $kg$ .

This representation allows us to generalize equation of the scalar moment of inertia for a fractal material

$$I^{(D)}(t) = \int_W \rho(\mathbf{r}, t) \mathbf{r}_{\perp}^2 dV_D, \quad (28)$$

where  $D$  is a mass dimension of fractal material.

### 3.3 Moment of inertia of fractal solid ball

Let us consider a fractal solid ball with radius  $R$ , and mass  $M$ . Note that the component of the radius perpendicular is

$$\mathbf{r}_{\perp}^2 = (r \sin \theta)^2.$$

Using the integration in a non-integer dimensional space, we have

$$I^{(D)} = \int_W d\mathbf{r} (r \sin \theta)^2 \rho(r, \theta) = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)} \int_0^\infty dr r^{D-1} \int_0^\pi d\theta \sin^{D-2} \theta (r \sin \theta)^2 \rho(r, \theta). \quad (29)$$

For homogeneous materials  $\rho(\mathbf{r}) = \rho_0$ , we get

$$\begin{aligned} I^{(D)} &= \frac{2\pi^{(D-1)/2} \rho_0}{\Gamma((D-1)/2)} \int_0^\infty dr r^{D+1} \int_0^\pi d\theta \sin^D \theta = \frac{2\pi^{(D-1)/2} \rho_0}{\Gamma((D-1)/2)} \frac{\pi^{1/2} \Gamma(D/2)}{\Gamma(D/2+1)} \frac{R^{D+2}}{D+2} = \\ &= \frac{2\pi^{D/2} \Gamma(D/2) \rho_0}{\Gamma(D/2-1/2) \Gamma(D/2+1)} \frac{R^{D+2}}{D+2} = \frac{\pi^{D/2} (D-1) \rho_0}{(D+2) \Gamma(D/2+1)} R^{D+2}, \end{aligned} \quad (30)$$

where we use

$$\int_0^\pi d\theta \sin^D \theta = \frac{\pi^{1/2} \Gamma(D/2)}{\Gamma(D/2+1)}. \quad (31)$$

Using the expression for mass (25), we can rewrite (30) as

$$I^{(D)} = \frac{2\Gamma(D/2)}{(D+2)\Gamma(D/2-1/2)} M_D R^2 = \frac{D-1}{D+2} M_D R^2, \quad (32)$$

where we use  $\Gamma(z) = (z-1)\Gamma(z-1)$ . For  $D = 3$ , equation (32) gives the well-known equation for moment of inertia of non-fractal ball  $I^3 = (2/5)MR^2$ .

## 4 Elasticity theory of fractal material

### 4.1 Elasticity theory of non-fractal material

The linear elastic constitutive relations for isotropic case is the well-known Hooke's law that has the form

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}, \quad (33)$$

where  $\lambda$  and  $\mu$  are the Lamé coefficients,  $\sigma_{ij}$  is the stress,  $\varepsilon_{kl}$  is the strain tensor. This expression determines the stress tensor in terms of the strain tensor for an isotropic material.

For homogenous and isotropic materials, the constitutive relation (33) gives the equation for the displacement vector fields  $\mathbf{u} = \mathbf{u}(\mathbf{r}, t)$  in the form

$$\lambda \operatorname{grad} \operatorname{div} \mathbf{u} + 2\mu \Delta \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u}, \quad (34)$$

where  $\mathbf{f} = \mathbf{f}(\mathbf{r}, t)$  is the vector field of external force density.

If the deformation in the material is described by  $\mathbf{u}(\mathbf{r}, t) = u(r, t) \mathbf{e}_r$ , then equation (34) has the form

$$(\lambda + 2\mu) \Delta u(r, t) + \mathbf{f}(r, t) = \rho D_t^2 u(r, t). \quad (35)$$

A formal generalization of equations (35) for fractal material in the framework of non-integer dimensional models is

$$(\lambda + 2\mu) {}^V\Delta_r^D \mathbf{u}(r, t) + \mathbf{f}(r, t) = \rho D_t^2 \mathbf{u}(r, t), \quad (36)$$

where  ${}^V\Delta_r^D$  is defined by (4). Equation (36) described dynamics of displacement vector for fractal elastic materials.

## 4.2 Strain and stress in non-integer dimensional space

Any deformation can be represented as the sum of a pure shear and a hydrostatic compression. To do so for fractal materials, we can use the identity

$$\varepsilon_{kl} = \left( \varepsilon_{kl} - \frac{1}{D} \delta_{kl} \varepsilon_{ii} \right) + \frac{1}{D} \delta_{kl} \varepsilon_{ii}. \quad (37)$$

The first term on the right is a pure shear, since the sum of diagonal terms is zero. Here we use the equation  $\delta_{ii} = D$  for non-integer dimensional space (for details see Property 4 in Section 4.3 of [32]). The second term is a hydrostatic compression. For  $D = 3$ , equation (37) has the well-known form

$$\varepsilon_{kl} = \left( \varepsilon_{kl} - \frac{1}{3} \delta_{kl} \varepsilon_{ii} \right) + \frac{1}{3} \delta_{kl} \varepsilon_{ii}, \quad (38)$$

where  $\delta_{ii} = 3$  is used.

The stress tensor can be represented as

$$\sigma_{kl} = K \varepsilon_{ii} \delta_{kl} + 2\mu \left( \varepsilon_{kl} - \frac{1}{D} \delta_{kl} \varepsilon_{ii} \right), \quad (39)$$

where the bulk modulus (modulus of hydrostatic compression) is related to the Lamé coefficients by

$$K = \lambda + \frac{2\mu}{D}. \quad (40)$$

In the hydrostatic compression of a body, the stress tensor is

$$\sigma_{kl} = -p \delta_{kl}. \quad (41)$$

Hence we have

$$\sigma_{kk} = -p D. \quad (42)$$

If we use the Hooke's law for isotropic case in the form, then

$$\sigma_{ii} = (\lambda D + 2\mu) \varepsilon_{ii}. \quad (43)$$

The components of the strain tensor is

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = (\mathbf{e}_r, \text{Grad}_r^D u_r). \quad (44)$$

Using (1), and the trace of the strain tensor

$$e(r) = Tr[\varepsilon_{kl}] = \varepsilon_{kk} = \text{Div}_r^D \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{D-1}{r} u_r. \quad (45)$$

Note that we can consider

$$e(r) - \varepsilon_{rr}(r) = \text{Div}_r^D \mathbf{u} - (\mathbf{e}_r, \text{Grad}_r^D u_r) = \frac{D-1}{r} u_r$$

as a sum of the angle diagonal components in the spherical coordinates of the strain tensor. For  $D = 3$ , we have the well-known sum of the angle diagonal components in the spherical coordinates of the strain tensor

$$\varepsilon_{\theta\theta} + \varepsilon_{\varphi\varphi} = \frac{2}{r} u_r.$$

When we consider the fractal medium distributed in three-dimensional space we can define the effective value of the diagonal angular components of the strain tensor

$$\varepsilon_{\theta\theta}^{eff} = \varepsilon_{\varphi\varphi}^{eff} = \frac{D-1}{2r} u_r. \quad (46)$$

Using (44) and (45), the components of the stress tensor  $\sigma_{kl} = \sigma_{kl}(r)$  in the spherical coordinates is

$$\sigma_{rr}(r) = 2\mu \varepsilon_{rr}(r) + \lambda e(r) = (2\mu + \lambda) \frac{\partial u_r}{\partial r} + \lambda \frac{D-1}{r} u_r. \quad (47)$$

It is well-known that the diagonal angular components of the stress tensor for  $D = 3$  in spherical coordinates are

$$\sigma_{\theta\theta}(r) = 2\mu \varepsilon_{\theta\theta}(r) + \lambda e(r), \quad \sigma_{\varphi\varphi}(r) = 2\mu \varepsilon_{\varphi\varphi}(r) + \lambda e(r). \quad (48)$$

For the fractal medium distributed in three-dimensional space we can define the effective value of the diagonal angular components of the stress tensor

$$\sigma_{\theta\theta}^{eff}(r) = 2\mu \varepsilon_{\theta\theta}^{eff}(r) + \lambda e(r), \quad (49)$$

$$\sigma_{\varphi\varphi}^{eff}(r) = 2\mu \varepsilon_{\varphi\varphi}^{eff}(r) + \lambda e(r). \quad (50)$$

Using the form of the effective components (46), we get

$$\sigma_{\theta\theta}^{eff}(r) = \sigma_{\varphi\varphi}^{eff}(r) = \lambda \frac{\partial u_r}{\partial r} + (\lambda + \mu) \frac{D-1}{r} u_r. \quad (51)$$

This equation define the diagonal angular components of the stress tensor for spherical coordinates.

### 4.3 Equilibrium equation for fractal materials

For static case, equation (36), we have

$${}^V\Delta_r^D \mathbf{u}(r) + (\lambda + 2\mu)^{-1} \mathbf{f}(r) = 0, \quad (52)$$

where  $\mathbf{u} = u_r \mathbf{e}_r$  and  $\mathbf{f} = f(r) \mathbf{e}_r$ . Here  $\lambda$  and  $\mu$  are the Lamé coefficients. Using (4), we represent equation (52) in the form

$$\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-1}{r} \frac{\partial u_r(r)}{\partial r} - \frac{D-1}{r^2} u_r(r) + (\lambda + 2\mu)^{-1} f(r) = 0. \quad (53)$$

The solution of (53) is

$$u_r(r) = C_1 r + C_2 r^{1-D} + I_f(D, r), \quad (54)$$

where  $C_1$  and  $C_2$  are constants defined by boundary conditions, and

$$I_f(D, r) = \frac{r}{D(\lambda + 2\mu)} \left( \frac{1}{r^D} \int dr r^D f(r) - \int dr r f(r) \right). \quad (55)$$

For  $f(r) = f_0$ , we get

$$I_f(D, r) = -\frac{f_0 r^2}{2(D+1)(\lambda + 2\mu)}, \quad (56)$$

and the displacement is

$$u_r(r) = C_1 r + C_2 r^{1-D} - \frac{f_0 r^2}{(D+1)(\lambda + 2\mu)}. \quad (57)$$

The components of the stress tensor  $\sigma_{kl} = \sigma_{kl}(r)$  for the spherical coordinates can be calculated by equations (47) and (51).

### 4.4 Elasticity of fractal hollow ball with inside and outside pressures

Let us determine the deformation of a hollow fractal ball with internal radius  $R_1$  and external radius  $R_2$  with the pressure  $p_1$  inside and the pressure  $p_2$  outside.

We can use the spherical polar coordinates with the origin at the center of the ball. The displacement vector  $\mathbf{u}$  is everywhere radial, and it is a function of  $r = |\mathbf{r}|$  alone. Then the equilibrium equation for fractal ball is

$$(\lambda + 2\mu) {}^V\Delta_r^D \mathbf{u}(r) = 0, \quad (58)$$

where  $\mathbf{u} = u_r \mathbf{e}_r$ . Using (4), we represent equation (52) in the form

$$\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-1}{r} \frac{\partial u_r(r)}{\partial r} - \frac{D-1}{r^2} u_r(r) = 0. \quad (59)$$

The solution of (53) is

$$u_r(r) = C_1 r + C_2 r^{1-D}. \quad (60)$$

The constants  $C_1$  and  $C_2$  are determined from the boundary conditions for radial stress

$$\sigma_{rr}(R_1) = -p_1, \quad \sigma_{rr}(R_2) = -p_2. \quad (61)$$

Using that the radial components of the stress is

$$\sigma_{rr}(r) = (2\mu + \lambda) \frac{\partial u_r}{\partial r} + \lambda \frac{D-1}{r} u_r, \quad (62)$$

we get

$$\sigma_{rr}(r) = (2\mu + D\lambda) C_1 + 2(1-D)\mu C_2 r^{-D}. \quad (63)$$

Then we have

$$(2\mu + D\lambda) C_1 + 2(1-D)\mu R_1^{-D} C_2 = -p_1, \quad (64)$$

$$(2\mu + D\lambda) C_1 + 2(1-D)\mu R_2^{-D} C_2 = -p_2. \quad (65)$$

As a result, the coefficients have the form

$$C_1 = \frac{-(p_1 R_2^{-D} - p_2 R_1^{-D})}{(2\mu + D\lambda)(R_2^{-D} - R_1^{-D})} = \frac{-(p_2 R_2^D - p_1 R_1^D)}{(2\mu + D\lambda)(R_2^D - R_1^D)}, \quad (66)$$

$$C_2 = \frac{-(p_2 - p_1)(R_1 R_2)^D}{2(1-D)\mu(R_2^{-D} - R_1^{-D})} = \frac{p_2 - p_1}{2(1-D)\mu(R_2^D - R_1^D)}. \quad (67)$$

Then the radial components of the stress is

$$\sigma_{rr}(r) = \frac{-(p_2 R_2^D - p_1 R_1^D)}{R_2^D - R_1^D} + \frac{(p_2 - p_1)(R_1 R_2)^D}{R_2^D - R_1^D} r^{-D}. \quad (68)$$

The stress distribution in a ball with pressure  $p_1 = p$  inside and  $p_2 = 0$  outside is gives by

$$\sigma_{rr}(r) = \frac{p R_1^D}{R_2^D - R_1^D} - \frac{p}{R_2^D - R_1^D} r^{-D} = \frac{p R_1^D}{R_2^D - R_1^D} \left( 1 - \left( \frac{R_2}{r} \right)^D \right). \quad (69)$$

The stress distribution in an infinite elastic material with spherical cavity with radius  $R$  subjected to hydrostatic compression

$$\sigma_{rr}(r) = -p \left( 1 - \left( \frac{R}{r} \right)^D \right). \quad (70)$$

that can be obtained by putting  $R_1 = R$ ,  $R_2 \rightarrow \infty$ ,  $p_1 = 0$  and  $p_2 = p$  in equation (68).

## 5 Elasticity of fractal material with radial distribution in cylinder and pipe

### 5.1 Elasticity of fractal pipe and cylinder

If we use the cylindrical coordinates with the  $z$ -axis and the vector field  $\mathbf{u}(\mathbf{r}, t)$  is a purely radial

$$\mathbf{u} = u_r(r) \mathbf{e}_r, \quad (71)$$

where  $\mathbf{e}_r = \mathbf{r}/r$ , then

$${}^V\Delta_r^D \mathbf{u} = \left( \frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-2}{r} \frac{\partial u_r}{\partial r} - \frac{D-2}{r^2} u_r(r) \right) \mathbf{e}_r. \quad (72)$$

Note that we have  $(D-2)$  instead of  $(D-1)$  in this equation. For  $D=3$ , equation (72) gives the well-known equation for elasticity of cylinder and pipe.

If  $\mathbf{u} = u_r(r) \mathbf{e}_r$  is the displacement vector for non-fractal materials in 3-dimensional case ( $D=3$ ), then the strain tensor  $\varepsilon_{ij}(\mathbf{r})$  has the following nonzero components that can be defined by

$$\varepsilon_{rr} = (\mathbf{e}_r, \text{grad } u_r) = \frac{\partial u_r}{\partial r}, \quad (73)$$

$$\varepsilon_{\varphi\varphi} = \text{div } \mathbf{u} - (\mathbf{e}_r, \text{grad } u_r) = \frac{u_r}{r}, \quad (74)$$

and the invariant

$$e = \varepsilon_{kk} = \text{div } \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r}. \quad (75)$$

These equations with divergence and gradient can be generalized for non-integer dimensional case for  $\mathbf{u} = u_r(r) \mathbf{e}_r$ .

For non-integer dimensional model of fractal materials, we can use the definitions for the components of displacement vector in non-integer dimensional space in the form

$$\varepsilon_{rr} = (\mathbf{e}_r, \text{Grad}_r^D u_r), \quad (76)$$

$$\varepsilon_{\varphi\varphi} = \text{Div}_r^D \mathbf{u} - (\mathbf{e}_r, \text{Grad}_r^D u_r), \quad (77)$$

where we use the invariant

$$e = \varepsilon_{kk} = \text{Div}_r^D \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{D-2}{r} u_r. \quad (78)$$

As a result, we get

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad (79)$$

$$\varepsilon_{\varphi\varphi} = \frac{D-2}{r} u_r. \quad (80)$$

Let us assume that the elastic constitutive relations for fractal material in isotropic case has the usual form

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}. \quad (81)$$

In this case, the non-zero components of stress tensor are

$$\sigma_{rr} = 2\mu \varepsilon_{rr} + \lambda e = (2\mu + \lambda) \frac{\partial u_r}{\partial r} + \lambda \frac{D-2}{r} u_r, \quad (82)$$

$$\sigma_{\varphi\varphi} = 2\mu \varepsilon_{\varphi\varphi} + \lambda e = \lambda \frac{\partial u_r}{\partial r} + (2\mu + \lambda) \frac{D-2}{r} u_r, \quad (83)$$

$$\sigma_{zz} = 2\mu \varepsilon_{zz} + \lambda e = \lambda \frac{\partial u_r}{\partial r} + \lambda \frac{D-2}{r} u_r, \quad (84)$$

where we use  $\varepsilon_{zz} = 0$ . For  $D = 3$ , we have the usual constitutive relations for isotropic case in cylindrical coordinates.

## 5.2 Elasticity of cylindrical fractal pipe with inside and outside pressures

Let us consider the deformation of a fractal solid cylindrical pipe with internal radius  $R_1$  external radius  $R_2$  with a inside pressure  $p_1$  and outside pressure  $p_2$ . We use the cylindrical coordinates with the  $z$ -axis along the axis of the pipe. When the pressure is uniform along the pipe, the deformation is a purely radial displacement  $\mathbf{u} = u_r(r) \mathbf{e}_r$ , where  $\mathbf{e}_r = \mathbf{r}/r$ . The equation for the displacement  $u_r(r)$  in fractal pipe is

$$\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-2}{r} \frac{\partial u_r}{\partial r} - \frac{D-2}{r^2} u_r = 0, \quad (85)$$

where  $0 < D \leq 3$ . If  $D = 3$ , we get the usual (non-fractal) case.

The general solution of equation (85), where  $D \neq 1$  and  $D \neq 2$ , has the form

$$u_r(r) = C_1 r + C_2 r^{2-D}. \quad (86)$$

Equations (85) with  $D = 1$  has the general solution

$$u_r(r) = C_1 r + C_2 r \ln(r). \quad (87)$$

For  $D = 2$ , equations (85) has the solution

$$u_r(r) = C_1 + C_2 r. \quad (88)$$

Note that that fractal dimension of the pipe material can be  $D = 1$  or  $D = 2$ . These cases do not correspond to the distribution of matter along the line and surface. These fractal dimensions describe a distribution of matter in 3-dimensional space (in the volume of pipe) such that the mass dimension is equal to  $D$ .



The constants  $C_1$  and  $C_2$  are determined by boundary conditions. Using that inside pressure is  $p_1$  and outside pressure is  $p_2$ , we get the boundary condition in the form

$$\sigma_{rr}(R_1) = -p_1, \quad \sigma_{rr}(R_2) = -p_2. \quad (89)$$

Using (86), we get

$$\frac{\partial u_r}{\partial r} = C_1 + (2 - D) C_2 r^{1-D}, \quad (90)$$

$$\frac{D-2}{r} u_r = (D-2) C_1 + (D-2) C_2 r^{1-D}. \quad (91)$$

Then

$$\sigma_{rr} = (2\mu + \lambda) \frac{\partial u_r}{\partial r} + \lambda \frac{D-2}{r} u_r = (2\mu + \lambda(D-1)) C_1 - 2\mu(D-2) C_2 r^{1-D}. \quad (92)$$

The boundary condition (89) has the form

$$(2\mu + \lambda(D-1)) C_1 - 2\mu(D-2) C_2 R_1^{1-D} = -p_1, \quad (93)$$

$$(2\mu + \lambda(D-1)) C_1 - 2\mu(D-2) C_2 R_2^{1-D} = -p_2. \quad (94)$$

As a result, we have

$$C_1 = -\frac{p_1 R_2^{1-D} - p_2 R_1^{1-D}}{(2\mu + \lambda(D-1)) (R_2^{1-D} - R_1^{1-D})}, \quad (95)$$

$$C_2 = \frac{p_2 - p_1}{2\mu(D-2) (R_2^{1-D} - R_1^{1-D})}. \quad (96)$$

The stress is

$$\sigma_{rr} = -\frac{p_1 R_2^{1-D} - p_2 R_1^{1-D}}{(R_2^{1-D} - R_1^{1-D})} - \frac{p_2 - p_1}{(R_2^{1-D} - R_1^{1-D})} r^{1-D}. \quad (97)$$

If  $2 < D < 3$  or  $1 < D < 2$ , then we can rewrite equation (97) in the form

$$\sigma_{rr} = \frac{p_1 R_1^{D-1} - p_2 R_2^{D-1}}{(R_2^{D-1} - R_1^{D-1})} - \frac{p_2 - p_1}{(R_2^{D-1} - R_1^{D-1})} \left( \frac{R_1 R_2}{r} \right)^{D-1}. \quad (98)$$

For the boundary conditions  $\sigma_{rr}(R_2) = 0$  and  $\sigma_{rr}(R_1) = -p$ , i.e.  $p_2 = 0$  and  $p_1 = p$  for (98), we have the solution of the form

$$\sigma_{rr} = \frac{p R_1^{D-1}}{(R_2^{D-1} - R_1^{D-1})} \left( 1 - \left( \frac{R_2}{r} \right)^{D-1} \right). \quad (99)$$

This is the deformation of cylindrical pipe with a pressure  $p$  inside and no pressure outside. For  $(D = 3)$  we have

$$\sigma_{rr} = \frac{p R_1^2}{(R_2^2 - R_1^2)} \left( 1 - \left( \frac{R_2}{r} \right)^2 \right) \quad (100)$$

that describes the stress of non-fractal material of pipe.

### 5.3 Rotating cylindrical fractal pipe

Let us consider the deformation of a fractal solid that is describes by equation with external force  $f(r)$  for the displacement field  $u_r(r)$  in fractal pine

$$\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-2}{r} \frac{\partial u_r}{\partial r} - \frac{D-2}{r^2} u_r + \frac{1}{\lambda+2\mu} f(r) = 0, \quad (101)$$

where  $D > 0$ . The general solution of equation (101) has the form

$$u_r(r) = C_1 r + C_2 r^{2-D} - \frac{1}{(D-1)(\lambda+2\mu)} \left( \int_{R_1}^{R_2} f(r) r dr - r^{2-D} \int_{R_1}^{R_2} f(r) r^{D-1} dr \right). \quad (102)$$

Equations (101) with  $D = 1$  has the general solution

$$u_r(r) = C_1 r + C_2 r \ln(r) + \frac{r}{\lambda+2\mu} \left( \int_{R_1}^{R_2} f(r) \ln(r) dr + \ln(r) \int_{R_1}^{R_2} f(r) dr \right). \quad (103)$$

For  $D = 2$ , equations (101) has the solution

$$u_r(r) = C_1 + C_2 r - \frac{1}{\lambda+2\mu} \left( \int_{R_1}^{R_2} f(r) r dr - r \int_{R_1}^{R_2} f(r) dr \right). \quad (104)$$

Let us consider the deformation of a fractal solid cylindrical pipe with internal radius  $R_1$  external radius  $R_2$  rotating uniformly about its axis with angular velocity  $\omega$ . Then the density of the centrifugal force is

$$f_r(r) = \rho_0 \omega^2 r. \quad (105)$$

We use the cylindrical coordinates with the  $z$ -axis along the axis of the cylinder. When the pressure is uniform along the pipe, the deformation is a purely radial displacement  $\mathbf{u} = u_r(r) \mathbf{e}_r$ , where  $\mathbf{e}_r = \mathbf{r}/r$ . The equation for the displacement  $u_r(r)$  in fractal material is

$$\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-2}{r} \frac{\partial u_r}{\partial r} - \frac{D-2}{r^2} u_r = -\frac{\rho_0 \omega^2}{\lambda+2\mu} r. \quad (106)$$

The general solution of equation (106) has the form

$$u_r(r) = C_1 r + C_2 r^{2-D} - A r^3, \quad (107)$$

where

$$A = \frac{\rho_0 \omega^2}{2(D+1)(\lambda+2\mu)}. \quad (108)$$

Using the condition that external forces do not act inside and outside the fractal pipe, we have the boundary condition

$$\sigma_{rr}(R_1) = 0, \quad \sigma_{rr}(R_2) = 0. \quad (109)$$

Using (107), we get

$$\frac{\partial u_r}{\partial r} = C_1 + (2 - D) C_2 r^{1-D} - 3 A r^2, \quad (110)$$

$$\frac{D-2}{r} u_r = (D-2) C_1 + (D-2) C_2 r^{1-D} - A (D-2) r^2. \quad (111)$$

Then

$$\begin{aligned} \sigma_{rr} &= (2\mu + \lambda) \frac{\partial u_r}{\partial r} + \lambda \frac{D-2}{r} u_r = \\ &= (2\mu + \lambda (D-1)) C_1 - 2\mu (D-2) C_2 r^{1-D} - A (6\mu + \lambda (D+1)) r^2. \end{aligned} \quad (112)$$

The boundary condition (109) has the form

$$(2\mu + \lambda (D-1)) C_1 - 2\mu (D-2) C_2 R_1^{1-D} = A (6\mu + \lambda (D+1)) R_1^2, \quad (113)$$

$$(2\mu + \lambda (D-1)) C_1 - 2\mu (D-2) C_2 R_2^{1-D} = A (6\mu + \lambda (D+1)) R_2^2. \quad (114)$$

Then

$$C_1 = \frac{A (6\mu + \lambda (D+1)) (R_2^{D+1} - R_1^{D+1})}{(2\mu + \lambda (D-1)) (R_2^{D-1} - R_1^{D-1})}, \quad (115)$$

$$C_2 = \frac{A (6\mu + \lambda (D+1)) (R_2^2 - R_1^2) (R_1 R_2)^{D-1}}{2\mu (D-2) (R_2^{D-1} - R_1^{D-1})}. \quad (116)$$

Substitution of (115) and (116) into (107)

$$\begin{aligned} u_r(r) &= \frac{A (6\mu + \lambda (D+1)) (R_2^{D+1} - R_1^{D+1})}{(2\mu + \lambda (D-1)) (R_2^{D-1} - R_1^{D-1})} r + \\ &+ \frac{A (6\mu + \lambda (D+1)) (R_2^2 - R_1^2) (R_1 R_2)^{D-1}}{2\mu (D-2) (R_2^{D-1} - R_1^{D-1})}, r^{2-D} - A r^3. \end{aligned} \quad (117)$$

where  $A$  is defined by (108).

For the fractal cylinder ( $R_1 = 0$ ,  $R_2 = R$ ), we have

$$u_r(r) = \frac{\rho_0 \omega^2}{2(D+1)(\lambda + 2\mu)} \left( \frac{6\mu + \lambda (D+1)}{2\mu + \lambda (D-1)} R^2 r - r^3 \right). \quad (118)$$

For  $D = 3$ , equation (118) gives

$$u_r(r) = \frac{\rho_0 \omega^2}{8(\lambda + 2\mu)} \left( \frac{3\mu + 2\lambda}{\mu + \lambda} R^2 r - r^3 \right). \quad (119)$$

Equation (119) describes the displacement field for elastic cylinder with non-fractal material.

## 6 Gradient elasticity model for fractal materials

### 6.1 Gradient elasticity theory of non-fractal material

In the papers [63, 64, 65] suggested to generalize the constitutive relations (33) by the gradient modification that contains the Laplacian in the form

$$\sigma_{ij} = \left( \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \right) - l_s^2 \Delta \left( \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \right), \quad (120)$$

where  $l_s$  is the scale parameter [66].

For homogenous and isotropic materials equation for (120) has the form

$$\lambda \left( 1 \pm l_s^2 \Delta \right) \text{grad div } \mathbf{u} + 2\mu \left( 1 \pm l_s^2 \Delta \right) \Delta \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u}, \quad (121)$$

where  $\mathbf{f}$  is the vector field of external force density.

Using relation

$$\text{grad div } \mathbf{u} = \text{curl curl } \mathbf{u} + \Delta \mathbf{u}, \quad (122)$$

we can rewrite equation (121) as

$$\lambda \left( 1 \pm l_s^2 \Delta \right) \text{curl curl } \mathbf{u} + (\lambda + 2\mu) \left( 1 \pm l_s^2 \Delta \right) \Delta \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u}. \quad (123)$$

If we assume that the displacement vector  $\mathbf{u}$  is everywhere radial and it is a function of  $r = |\mathbf{r}|$  alone ( $u_k = u_k(|\mathbf{r}|)$ ), then

$$\text{curl } \mathbf{u} = 0.$$

As a result, equation (123) has the form

$$(\lambda + 2\mu) \left( 1 \pm l_s^2 \Delta \right) \Delta \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u}. \quad (124)$$

This is gradient elasticity equation for homogenous and isotropic materials with the spherical symmetry. For the non-gradient model ( $l_s^2 = 0$ ), equation (124) for the displacement gives (35).

### 6.2 Gradient elasticity of fractal material

A formal generalization of equations (124) for fractal material in the framework of continuum models with non-integer dimensional space, where the displacement vector  $\mathbf{u} = \mathbf{u}(r, t)$ , does not depend on the angles, has the form

$$(\lambda + 2\mu) \left( 1 \pm l_s^2(D)^V \Delta_r^D \right)^V \Delta_r^D \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u}. \quad (125)$$

This is fractional gradient elasticity equation for homogenous and isotropic materials with the spherical symmetry. Let us consider equation (125) for static case ( $D_t^2 \mathbf{u} = 0$ ) with a minus in front of Laplacian, i.e. the GRADELA model for fractal materials [66],

$$(\lambda + 2\mu) \left( 1 - l_s^2(D)^V \Delta_r^D \right)^V \Delta_r^D \mathbf{u} + \mathbf{f} = 0. \quad (126)$$

We can rewrite this equation as

$$({}^V\Delta_r^D)^2 \mathbf{u} - l_s^{-2}(D) {}^V\Delta_r^D \mathbf{u} - (\lambda + 2\mu)^{-1} l_s^{-2}(D) \mathbf{f} = 0. \quad (127)$$

Using the vector Laplacian (4), we have

$${}^V\Delta_r^D \mathbf{u}(r) = \left( \frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-1}{r} \frac{\partial u_r(r)}{\partial r} - \frac{D-1}{r^2} u_r(r) \right) \mathbf{e}_r. \quad (128)$$

and

$$\begin{aligned} ({}^V\Delta_r^D)^2 \mathbf{u}(r) = & \left( \frac{\partial^4 u_r(r)}{\partial r^4} + \frac{2(D-1)}{r} \frac{\partial^3 u_r(r)}{\partial r^3} + \right. \\ & \left. + \frac{(D-1)(D-5)}{r^2} \frac{\partial^2 u_r(r)}{\partial r^2} - \frac{3(D-1)(D-3)}{r^3} \frac{\partial u_r(r)}{\partial r} + \frac{3(D-1)(D-3)}{r^4} u_r(r) \right) \mathbf{e}_r. \end{aligned} \quad (129)$$

Using (128 - 129) and  $\mathbf{f}(r) = f(r) \mathbf{e}_r$ , equation (127) gives

$$\begin{aligned} & \frac{\partial^4 u_r(r)}{\partial r^4} + \frac{2(D-1)}{r} \frac{\partial^3 u_r(r)}{\partial r^3} + \left( \frac{(D-1)(D-5)}{r^2} - l_s^{-2}(D) \right) \frac{\partial^2 u_r(r)}{\partial r^2} - \\ & - \left( \frac{3(D-1)(D-3)}{r^3} + l_s^{-2}(D) \frac{D-1}{r} \right) \frac{\partial u_r(r)}{\partial r} + \\ & + \left( \frac{3(D-1)(D-3)}{r^4} + l_s^{-2}(D) \frac{D-1}{r^2} \right) u_r(r) - (\lambda + 2\mu)^{-1} l_s^{-2}(D) f(r) = 0. \end{aligned} \quad (130)$$

General solution for the case  $f(r) = 0$  is

$$u_r(r) = C_1 r + C_2 r^{1-D} - C_3 I_I(D, r) - C_4 I_K(D, r), \quad (131)$$

where  $I_I(D, r)$  and  $I_K(D, r)$  are the integrals of the Bessel functions

$$I_I(D, r) = D r \int dr r^{-D-1} \int dr r^{D/2+1} I_{D/2}(r/l_s(D)), \quad (132)$$

$$I_K(D, r) = D r \int dr r^{-D-1} \int dr r^{D/2+1} K_{D/2}(r/l_s(D)), \quad (133)$$

where  $I_\alpha(x)$  and  $K_\alpha(x)$  are Bessel functions of the first and second kinds.

## 7 Thermoelasticity of fractal material

Let us consider a generalization of thermoelasticity [67, 68, 69] for fractal material. In this section we consider a non-integer-dimensional model of thermoelasticity of fractal material. Note that thermoelasticity of fractal materials in the framework of the fractional-integral continuum model [10, 9, 21], has been considered by Ostoja-Starzewski in [70, 71].

## 7.1 Thermoelastic constitutive relation for fractal material

If the isotropic material is non-uniformly heated, then the constitutive relation for a thermoelastic material must include [62] the term

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} - K \alpha (T - T_0) \delta_{ij}, \quad (134)$$

where  $\lambda$  and  $\mu$  are the Lamé coefficients,  $K$  is the bulk modulus or modulus of compression. The third term in equation (134) gives the additional stresses caused by the change in temperature.

The bulk modulus for fractal materials is related to the Lamé coefficients by

$$K = \lambda + \frac{2}{D}\mu. \quad (135)$$

In this formula, we use the dimension  $D$  instead of 3 because  $\delta_{kk} = D$  for non-integer dimensional space (see Property 4 in Section 4.3 of [32]).

If external forces being absent, then the stress is equal to zero  $\sigma_{ij} = 0$  and we have a free thermal expansion. Using  $\sigma_{ij} = 0$ , equation (134) gives

$$\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} - K \alpha (T - T_0) \delta_{ij} = 0. \quad (136)$$

Using  $\delta_{kk} = D$  and (135), we obtain

$$\varepsilon_{kk} = \alpha (T - T_0). \quad (137)$$

Because the function  $e = \varepsilon_{kk}$  describes the relative change of volume caused by deformation, then  $\alpha$  is the thermal expansion coefficient of the material [62].

The constitutive relation (134) for fractal material can be represented in the form

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} - (D\lambda + 2\mu) \alpha_T (T - T_0) \delta_{ij}, \quad (138)$$

where we use the dimension  $D$  instead of 3 since  $K = (D\lambda + 2\mu)/D$ .

## 7.2 Thermoelastic equations for fractal material

For homogenous and isotropic non-fractal materials, we have the thermoelasticity equation

$$\lambda \operatorname{grad} \operatorname{div} \mathbf{u} + 2\mu \Delta \mathbf{u} + \mathbf{f} - (3\lambda + 2\mu) \alpha \operatorname{grad} T = \rho D_t^2 \mathbf{u}, \quad (139)$$

where  $\mathbf{f}$  is the external force density vector field. For the case  $\mathbf{u} = \mathbf{u}(r, t)$  and  $\mathbf{f} = \mathbf{f}(r, t)$ , equation (139) gives

$$(\lambda + 2\mu) \Delta \mathbf{u} + \mathbf{f} - (3\lambda + 2\mu) \alpha \operatorname{grad} T = \rho D_t^2 \mathbf{u}. \quad (140)$$

The thermoelasticity equation for fractal materials in the framework of models with non-integer dimensional space has the form

$$(\lambda + 2\mu) {}^V \Delta_r^D \mathbf{u}(r, t) + \mathbf{f}(r, t) - (D\lambda + 2\mu) \alpha \operatorname{Grad}_r^D T(r, t) = \rho D_t^2 \mathbf{u}(r, t), \quad (141)$$

where we assume  $\mathbf{f} = f_r(r, t) \mathbf{e}_r$  and  $\mathbf{u} = u_r(r, t) \mathbf{e}_r$ . Using (4), equation (141) can be represented in the form

$$\begin{aligned} & \frac{\partial^2 u_r(r, t)}{\partial r^2} + \frac{D-1}{r} \frac{\partial u_r(r, t)}{\partial r} - \frac{D-1}{r^2} u_r(r, t) + \\ & + \frac{1}{\lambda + 2\mu} f_r(r, t) - \frac{\alpha (D\lambda + 2\mu)}{\lambda + 2\mu} \frac{\partial T(r, t)}{\partial r} = \frac{\rho}{\lambda + 2\mu} D_t^2 u_r(r, t). \end{aligned} \quad (142)$$

If the fractal material is non-uniformly heated, then the equation of equilibrium has the form

$$\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-1}{r} \frac{\partial u_r(r)}{\partial r} - \frac{D-1}{r^2} u_r(r) = \frac{(D\lambda + 2\mu)\alpha}{\lambda + 2\mu} \frac{\partial T(r)}{\partial r}. \quad (143)$$

This is thermoelasticity equation in spherical coordinates for pure radial deformation of fractal materials with fractal dimension  $D$ . For  $D = 3$ , we get the usual equation for thermoelasticity of solid ball [62].

General solution of equation (143) has the form

$$u_r(r) = C_1 r + \frac{c_2}{r^{D-1}} + \frac{(D\lambda + 2\mu)\alpha}{D(\lambda + 2\mu)} \left( r T(r) - \frac{D}{r^{D-1}} \int r^D T(r) dr \right), \quad (144)$$

where  $C_1$  and  $C_2$  are defined by boundary conditions.

## 8 Conclusion

In this paper, continuum models with non-integer dimensional space are suggested to describe isotropic fractal materials. A generalization of differential vector operators for non-integer dimensional space is proposed to describe elasticity of fractal materials in the framework of continuum models. The differential operators of first and second orders for non-integer dimensional space are suggested for rotationally covariant scalar and vector functions. We consider applications for elasticity theory in the case of spherical and axial symmetries of the fractal material. Elastic properties of fractal hollow ball and cylindrical fractal pipe with inside and outside pressures, rotating cylindrical fractal elastic pipe are described. Equations for thermoelasticity and gradient elasticity of fractal materials are solved.

Although the non-integer dimension does not reflect all properties of the fractal material, the suggested models with non-integer dimensional space nevertheless allows us to derive a number of important conclusions about the behavior of fractal materials. Therefore continuum models with non-integer dimensional spaces can be successfully used to describe elasticity and thermoelasticity of fractal materials.

The proposed continuum models of fractal materials can be extended to more complex fractal materials: (1) We assume that continuum models with non-integer dimensional space can be generalized for anisotropic fractal materials [74]; (2) The non-integer dimensional models of fractal elastic materials can easily be generalized for the boundary dimensions  $d \neq D - 1$

[75]. (3) We also assume that differential and integral operators of fractional orders can also be defined for non-integer dimensional spaces to take into account non-locality of fractal materials. Note that dimensional continuation of the Riesz fractional integrals and derivatives [72] to generalize differential and integrals of fractional orders for non-integer dimensional space has been considered in [73].

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